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1983 J. Phys. A: Math. Gen. 16 221

(http://iopscience.iop.org/0305-4470/16/2/006)

J. Phys. A: Math. Gen. 16 (1983) 221-236. Printed in Great Britain

Landau-Lifshitz equation: solitons, quasi-periodic solutions and infinite-dimensional Lie algebras

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Received 27 April 1982

Abstract. The hierarchy of the Landau-Lifshitz equation $\boldsymbol{S}_t = \boldsymbol{S}\times\boldsymbol{S}_{xx} + \boldsymbol{S}\times J\boldsymbol{S}$ with full anisotropy is formulated in terms of a free fermion $\phi(P)$ on an elliptic curve. An infinite-dimensional Lie algebra spanned by quadratic forms of $\phi(P)$ is shown to act on solutions as infinitesimal Bäcklund transformations. On the basis of a bilinear identity of wavefunctions, an N-soliton formula is proved and quasi-periodic solutions are constructed.

1. Introduction

The Landau-Lifshitz equation

$$\boldsymbol{S}_t = \boldsymbol{S}\times\boldsymbol{S}_{xx} + \boldsymbol{S}\times J\boldsymbol{S} \tag{1.1}$$

where

$$\boldsymbol{S} = (S_1, S_2, S_3), \qquad S_1^2 + S_2^2 + S_3^2 = 1,$$

$$J = \operatorname{diag}(J_1, J_2, J_3) \text{ a constant diagonal matrix,}$$

is a classical equation for nonlinear spin waves in a ferromagnet. An interesting feature of this equation is that it contains elliptic moduli.

Sklyanin (1979) and Borovik found the Lax pair

$$\partial W/\partial x_1 = LW, \qquad \partial W/\partial x_2 = MW$$

with $x_1 = x$, $x_2 = -\mathrm{i}t$ and

$$L = \sum_{\alpha=1}^{3} z_\alpha S_\alpha \sigma_\alpha,$$

$$M = \mathrm{i}\sum_{\alpha,\beta,\gamma=1}^{3} z_\alpha \sigma_\alpha S_\beta S_{\gamma x}\varepsilon^{\alpha\beta\gamma} + 2z_1z_2z_3\sum_{\alpha=1}^{3} z_\alpha^{-1} S_\alpha\sigma_\alpha$$

$$\left(\sigma_1 = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} & -\mathrm{i}\\ \mathrm{i} & \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & \\ & -1\end{pmatrix}\right). \tag{1.2}$$

Here the spectral parameters (z_1, z_2, z_3) constitute an algebraic coordinate of an elliptic curve $\tilde{E}$ defined by

$$z_\alpha^2 - z_\beta^2 = \tfrac{1}{4}(J_\alpha - J_\beta) \qquad (\alpha, \beta = 1, 2, 3). \tag{1.3}$$

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Bogdan and Kovalev (1980) found a two-soliton solution:

$$S_1 = \frac{f^*g + fg^*}{f^*f + g^*g}, \qquad S_2 = -\mathrm{i}\frac{f^*g - fg^*}{f^*f + g^*g}, \qquad S_3 = \frac{f^*f - g^*g}{f^*f + g^*g} \tag{1.4}$$

where

$$f = 1 + \exp(\xi_1 + \xi_2), \qquad f^* = 1 + c_{12}h_1h_2\exp(\xi_1 + \xi_2),$$

$$g = \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2}, \qquad g^* = h_1\,\mathrm{e}^{\xi_1} + h_2\,\mathrm{e}^{\xi_2},$$

$$\xi_j = \xi_j^0 + k_jx_1 + \omega_jx_2,$$

$$\omega_j^2 = (k_j^2 - a^2)(k_j^2 - b^2), \qquad a^2 = J_3 - J_1, \qquad b^2 = J_3 - J_2,$$

$$h_j = \frac{\omega_j + k_j^2 - a^2}{\omega_j - k_j^2 + a^2}, \qquad c_{12} = \frac{k_1 - k_2}{k_1 + k_2}\frac{\omega_1(k_2^2 - a^2) - \omega_2(k_1^2 - a^2)}{\omega_1(k_2^2 - a^2) + \omega_2(k_1^2 - a^2)}.$$

Here the momentum k and the energy ω constitute an algebraic coordinate of another elliptic curve E defined by

$$\omega^2 = (k^2 - a^2)(k^2 - b^2). \tag{1.5}$$

The curves $\tilde{E}$ and E are related by the two-sheeted unramified covering map:

$$\begin{array}{ccc} \pi : \tilde{E} & \longrightarrow & E \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ \tilde{P} = (z_1, z_2, z_3) & \longrightarrow & P = (k, \omega) \end{array}$$

$$k = 2z_3, \qquad \omega = 4z_1z_2.$$

If we introduce a Cauchy kernel of the curve E by

$$K(P, P') = \tfrac{1}{2}(\omega + k^2 - \omega' - k'^2)/(k + k') \tag{1.6}$$

the mysterious factor c_{12} in Bogdan and Kovalev's solution is neatly written as

$$c_{12} = [4/(a^2 - b^2)]K(P_1, P_2)K(P_2^{\#}, P_1^{\#}) \tag{1.7}$$

where $\#$ denotes the involution of E,

$$P^{\#} = (-k, -\omega).$$

In a series of papers (Kashiwara and Miwa 1981, Date *et al* 1981a, b, 1982a, b) a new method for solving soliton equations has been developed. The heart of the method is to recognise that the solution spaces of soliton equations in Hirota's bilinear form can be identified with orbits of infinite-dimensional groups. As the most appropriate language for the description of infinite-dimensional groups, free fermions are used. The principal aim of this paper is to extend the method so that the Landau–Lifshitz equation can also be handled. Guided by the observation (1.7), we introduce a free fermion $\phi(P)$ on the curve E with the vacuum expectation value

$$\langle \mathrm{vac}|\phi(P)\phi(P')|\mathrm{vac}\rangle = K(P, P'). \tag{1.8}$$

Then the complete integrability of the Landau–Lifshitz equation follows naturally, in the same manner as the previous cases, the Kadomtsev–Petviashvili equation, the Korteweg–de Vries equation, etc, where free fermions on the rational curve $\mathbb{P}^1$ were used.

This paper is organised as follows. In § 2, the free fermion $\phi(P)$ is introduced on the elliptic curve E, and the Fock representation of the operator algebra is constructed in terms of the so-called vertex operators. In § 3, the hierarchy of the Landau–Lifshitz and its higher-order equations is shown to be a reduction of the two-component theory of the fermion constructed in § 2. From this follows a bilinear identity, which is equivalent both to the linearisation (Sklyanin 1979) in the sense of Lax and to the bilinearisation (Bogdan and Kovalev 1980) in the sense of Hirota (1982). Explicit forms of N solitons and the corresponding wavefunctions are given. Infinitesimal Bäcklund transformations are shown to form an infinite-dimensional Lie algebra isomorphic to $\mathrm{sl}(2, \mathbb{C}[k, k^{-1}, \omega])\oplus\mathbb{C}\cdot 1$. In § 4, the rational limit $a, b \to 0$ is discussed from the viewpoint of operator theory. Finally, in § 5, by exploiting the bilinear identity, quasi-periodic solutions are constructed (cf Cherednik 1981).

2. Free fermion on a curve

We denote by $\infty_\pm$ on E the point at infinity with $(k, \omega) = (\infty, \infty)$ and $\omega/k^2 = \pm 1$. The Cauchy kernel we have chosen in (1.6) has its poles in P at $P = P'^{\#}$ and $P = \infty_+$, and its zeros in P at $P = P'$ and $P = \infty_-$. It is so normalised that

$$\int \mathrm{d}P\, K(P, P') = 1$$

where

$$\mathrm{d}P = \mathrm{d}k/2\pi \mathrm{i}\omega$$

and the integration is around $P = P'^{\#}$.

We denote by $\phi(P)$ $(P \in E)$ the free fermion with the vacuum expectation value (1.7). For a meromorphic function $\alpha(P)$ on E we define a fermion by

$$\phi[\alpha] = \int_{\infty_+ \cup \infty_-} \mathrm{d}P\, \alpha(P)\phi(P).$$

Here the integration contour is a curve on E which encircles ∞_+ and ∞_- in a clockwise direction (figure 1). The expectation value (1.7) should be understood as follows. The contour for $\mathrm{d}P$ integration is so chosen that $P'^{\#}$ is outside it, while the contour for $\mathrm{d}P'$ integration is so chosen that $P^{\#}$ is inside it. This means, in particular, that $K(P, P') + K(P', P)$ serves as the δ function with support at $P = P'^{\#}$ for $\mathrm{d}P$ integration or $\mathrm{d}P'$ integration.

If $\alpha(P)$ is regular on $E - \infty_+ - \infty_-$, the expectation value

$$\langle \mathrm{vac}|\phi(Q)\phi[\alpha]|\mathrm{vac}\rangle = \int_{\infty_+ \cup \infty_-} \mathrm{d}P\, \alpha(P) K(Q, P)$$

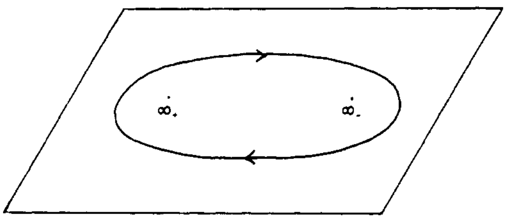

Figure 1. Contour of integration on E.

vanishes because of the absence of poles on the left bank of the contour for dP integration. Similar arguments lead us to the decomposition of the space of fermions into the annihilation part and the creation part: annihilation operators,

$$\phi[\alpha]|\text{vac}\rangle = 0$$

if and only if $\alpha(P)$ is regular on $E-\infty_+-\infty_-$, creation operators,

$$\langle\text{vac}|\phi[\alpha] = 0$$

if and only if $\alpha(P)$ has at most a simple pole at ∞_- and at least a simple zero at ∞_+.

As a basis of annihilation and creation operators we define

$$\psi_l = \sqrt{c}\int_{\infty_+\cup\infty_-} \mathrm{d}P\, k^{-l-1}\phi(P), \qquad \psi_l^* = \sqrt{c}\int_{\infty_+\cup\infty_-} \mathrm{d}P\, k^l h\phi(P^{\#}),$$

for $l\in\mathbb{Z}$. Here we have set $c=(a^2-b^2)/4$. They satisfy the anticommutation relations

$$[\psi_l, \psi_m]_+ = 0, \qquad [\psi_l^*, \psi_m^*]_+ = 0, \qquad [\psi_l, \psi_m^*]_+ = \delta_{lm}.$$

Moreover we have

$$\psi_l|\text{vac}\rangle = 0\ (l<0), \qquad \psi_l^*|\text{vac}\rangle = 0\ (l\geqslant 0),$$
$$\langle\text{vac}|\psi_l = 0\ (l\geqslant 0), \qquad \langle\text{vac}|\psi_l^* = 0\ (l<0).$$

The Majorana fermion is recovered by the formula

$$\phi(P) = \sqrt{c}\left(h\sum_{l\in\mathbf{Z}} k^l\psi_l + \sum_{l\in\mathbf{Z}}(-k)^{-l-1}\psi_l^*\right). \tag{2.1}$$

Thus the Clifford algebra generated by $\phi(P)$, which we denote by A, and the vacua are independent of the elliptic moduli, and they are identical with those employed in the study of the KP hierarchy (Date *et al* 1981a, b).

Now we introduce time evolutions. We set

$$\xi_n(x,P) = n\log h = \sum_{l\text{ odd}} x_l k^l + \sum_{l\text{ even}} x_l c^{1/2}(h^{l/2}-h^{-l/2}). \tag{2.2}$$

If we fix a branch of $\log h$, $\xi_n(x,P)$ is single valued on a contour on E of figure 1. Thus we can define a Hamiltonian by

$$H_n(x) = \tfrac{1}{2}\int_{\infty_+\cup\infty_-} \mathrm{d}P\,\xi_n(x,P){:}\,\phi(P)\phi(P^{\#}){:}, \tag{2.3}$$

which enjoys the properties

$$\exp[H_n(x)]\phi(P)\exp[-H_n(x)] = \exp[\xi_n(x,P)]\phi(P), \qquad H_n(x)|\text{vac}\rangle = 0.$$

Since

$$\int_{\infty_+\cup\infty_-} \mathrm{d}P{:}\,\phi(P)\phi(P^{\#}){:} = 0,$$

there is no ambiguity in the definition of $H_n(x)$ caused by the ambiguity of the branch of $\log h$.

The Fock space $A|\text{vac}\rangle$ can be realised by exploiting the time evolution induced by the Hamiltonian $H_n(x)$. Namely, as we shall see below, for $a\in A$, $a|\text{vac}\rangle = 0$ if and only if the expectation values

$$f_n(x\,;a)=\langle \mathrm{vac}|\exp[H_n(x)]a|\mathrm{vac}\rangle \tag{2.4}$$

and

$$g_n(x\,;a)=\langle \mathrm{vac}|\phi_0\exp[H_n(x)]a|\mathrm{vac}\rangle \tag{2.5}$$

vanish identically. In (2.5) we have set

$$\phi_0=\int_{\infty_+\cup\infty_-}\mathrm{d}P\,\phi(P).$$

In other words, the pair $(f_n(x\,;a), g_n(x,a))$ represents the vector $a|\mathrm{vac}\rangle$. For instance, by using (A4) and (A5), we have

$$\langle \mathrm{vac}|\exp[H_n(x)]\phi(P_1)\ldots\phi(P_N)|\mathrm{vac}\rangle$$

$$=\begin{cases}0 & (N\text{ odd}),\\ \exp\Big(\sum_{j=1}^{N}\xi_n(x,P_j)\Big)c^{-N(N-2)/4}\Big(\prod_{j=1}^{N}h(P_j)\Big)^{1-N/2}\prod_{1\le i,j\le N}K(P_i,P_j) & (N\text{ even}),\end{cases}$$

$$\langle \mathrm{vac}|\phi_0\exp[H_n(x)]\phi(P_1)\ldots\phi(P_N)|\mathrm{vac}\rangle$$

$$=\begin{cases}\exp\Big(\sum_{j=1}^{N}\xi_n(x,P_j)\Big)c^{-(N-1)^2/4}\Big(\prod_{j=1}^{N}h(P_j)\Big)^{(1-N)/2}\prod_{1\le i,j\le N}K(P_i,P_j) & (N\text{ odd}),\\ 0 & (N\text{ even}).\end{cases}$$

The action of $\phi(P)$ in $A|\mathrm{vac}\rangle$,

$$a|\mathrm{vac}\rangle\mapsto\phi(P)a|\mathrm{vac}\rangle,$$

can be realised in terms of a kind of vertex operator. To see this, first we introduce a shift operator $\mathscr{D}(P)$ for $P\in E$ near ∞_+ by

$$\mathscr{D}(P)=\exp\Big(-\sum_{l\text{ odd}}\frac{k^{-l}}{l}\frac{\partial}{\partial x_l}-\sum_{l\text{ even}}\frac{h^{-l/2}}{lc^{l/2}}\frac{\partial}{\partial x_l}\Big),$$

and for $P\in E$ near ∞_- by

$$\mathscr{D}(P)=\exp\Big(-\sum_{l\text{ odd}}\frac{k^{-l}}{l}\frac{\partial}{\partial x_l}+\sum_{l\text{ even}}\frac{h^{l/2}}{lc^{l/2}}\frac{\partial}{\partial x_l}+\frac{\partial}{\partial n}\Big).$$

The following are direct consequences of the definition.

$$\mathscr{D}(P)\exp[\xi_n(x,Q)]=\Big(\frac{K(P,Q)}{K(P,Q^{\#})}\Big)^{1/2}\exp[\xi_n(x,Q)],\qquad \mathscr{D}(P^{\#})\mathscr{D}(P)=\exp(\partial/\partial n). \tag{2.6}$$

By using (2.5), (A4) and (A5) we can show

$$\langle \mathrm{vac}|\exp[H_n(x)]\phi(P)=(ch(P))^{1/2}\exp[\xi_n(x,P)]\mathscr{D}(P)\langle \mathrm{vac}|\phi_0\exp[H_n(x)], \tag{2.7}$$

$$\langle \mathrm{vac}|\phi_0\exp[H_n(x)]\phi(P)=\exp[\xi_n(x,P)]\mathscr{D}(P)\exp(-\partial/\partial n)\langle \mathrm{vac}|\exp[H_n(x)]. \tag{2.8}$$

In (2.7) the branch of $(h(P))^{1/2}$ should be carefully chosen:

$$(h(P))^{1/2}=k/\sqrt{c}+\dots \qquad \text{for } P=(k, k^2[(1-a^2/k^2)(1-b^2/k^2)]^{1/2}),$$
$$(h(P))^{1/2}=-\sqrt{c}/k+\dots \qquad \text{for } P=(k, -k^2[(1-a^2/k^2)(1-b^2/k^2)]^{1/2}).$$

Thus we have shown that the action of $\phi(P)$ is realised by the following 2×2 linear differential operator of infinite order:

$$\begin{pmatrix} f_n(x;\phi(P)a) \\ g_n(x;\phi(P)a)\end{pmatrix} = \begin{pmatrix} 0 & (ch(P))^{1/2}\exp[\xi_n(x,P)]\mathscr{D}(P) \\ \exp[\xi_n(x,P)]\mathscr{D}(P)\exp(-\partial/\partial n) & 0 \end{pmatrix} \times \begin{pmatrix} f_n(x;a) \\ g_n(x;a)\end{pmatrix}.$$

In particular, if $f_n(x;a)=g_n(x;a)=0$, then $a|\text{vac}\rangle=0$.

The infinite-dimensional Lie algebra of quadratic elements in A can be realised in terms of scalar vertex operators on the space of $f_n(x)$. In fact, we have

$$\phi(P)\phi(Q)=K(P,Q)\exp[\xi_n(x,P)+\xi_n(x,Q)]\mathscr{D}(P)\mathscr{D}(Q)\exp(-\partial/\partial n). \tag{2.9}$$

The Heisenberg algebra $(\oplus_{l\geq 1}\mathbb{C}x_l)\oplus\mathbb{C}n\oplus(\oplus_{l\geq 1}\mathbb{C}\partial/\partial x_l)\oplus\mathbb{C}\partial/\partial n$ is contained in this Lie algebra. From (2.2) and (2.3) follows

$$\frac{\partial}{\partial x_l}=\begin{cases} \frac{1}{2}\int_{\infty_+\cup\infty_-} k^l\colon\phi(P)\phi(P^{*})\colon \mathrm{d}P, & l\geq 1, \text{odd}, \\ \frac{1}{2}\int_{\infty_+\cup\infty_-} c^{l/2}(h^{l/2}-h^{-l/2})\colon\phi(P)\phi(P^{*})\colon \mathrm{d}P, & l\geq 2, \text{even}, \end{cases} \tag{2.10}$$

$$\frac{\partial}{\partial n}=\frac{1}{2}\int_{\infty_+\cup\infty_-}\log h\colon\phi(P)\phi(P^{*})\colon \mathrm{d}P.$$

In order to obtain the expressions for x_l and n, we specialise (2.9) to

$$\begin{aligned}
&\colon\phi(P)\phi(P^{*})\colon \\
&\quad=\lim_{P'\to P^{*}}(\phi(P)\phi(P')-K(P,P')) \\
&\quad=c(h-h^{-1})\sum_{l\text{ odd}}(lk^{l-1}x_l+k^{-l-1}\,\partial/\partial x_l) \\
&\qquad+k\sum_{l\text{ even}}[lc^{l/2}(h^{l/2}+h^{-l/2})x_l+(ch)^{-l/2}\,\partial/\partial x_l]+2nk.
\end{aligned}$$

From this follows

$$lx_l=\begin{cases} \int_{\infty_+} k^{-l}\colon\phi(P)\phi(P^{*})\colon \mathrm{d}P, & l\geq 1, \text{odd}, \\ \int_{\infty_+} (ch)^{-l/2}\colon\phi(P)\phi(P^{*})\colon \mathrm{d}P, & l\geq 2, \text{even}, \end{cases} \tag{2.11}$$

$$n=\frac{1}{2}\int_{\infty_+}\colon\phi(P)\phi(P^{*})\colon \mathrm{d}P.$$

3. Landau–Lifshitz hierarchy

The Landau–Lifshitz equation (1.1) was put into bilinear form by Hirota (1982)

$$\begin{aligned} &D_1(f^*\cdot f+g^*\cdot g)=0, \qquad (D_2-D_1^2)(f^*\cdot f-g^*\cdot g)=0,\\ &[D_2-D_1^2+\tfrac{1}{2}(a^2+b^2)]f^*\cdot g+\tfrac{1}{2}(a^2-b^2)g^*\cdot f=0, \\ &[D_2-D_1^2+\tfrac{1}{2}(a^2+b^2)]g^*\cdot f+\tfrac{1}{2}(a^2-b^2)f^*\cdot g=0. \end{aligned} \tag{3.1}$$

Here f, f^*, g and g^* are related to the spin variables S_α by (1.4). We are now going to solve (3.1) in terms of field operators.

Let $\phi^{(i)}(P)$ $(i=1,2)$ denote copies of the free fields introduced in § 2, and let $H_n^{(i)}(x)$ be the corresponding Hamiltonian. For an element g of the Clifford group generated by $\phi^{(i)}(P)$ $(i=1,2)$, we set

$$\tau_{n_1n_2}(x^{(1)},x^{(2)})=\langle\mathrm{vac}|e^{\mathrm{H}}g|\mathrm{vac}\rangle, \tag{3.2}$$

$$\begin{aligned} &\sigma_{n_1n_2}(x^{(1)},x^{(2)})=\langle\mathrm{vac}|\phi_0^{(1)}\phi_0^{(2)}\,e^{\mathrm{H}}g|\mathrm{vac}\rangle,\\ &w_{n_1n_2}^{(ij)}(x^{(1)},x^{(2)};P)=\langle\mathrm{vac}|\phi_0^{(i)}\,e^{\mathrm{H}}\phi^{(j)}(P)g|\mathrm{vac}\rangle \qquad (i,j=1,2), \end{aligned} \tag{3.3}$$

where $\mathrm{H}=H_{n_1}^{(1)}(x^{(1)})+H_{n_2}^{(2)}(x^{(2)})$. It can be shown that (3.2) and (3.3) provide respectively the τ functions and the wavefunctions for a certain hierarchy of nonlinear equations of KP type (cf Kashiwara and Miwa 1981, Date *et al* 1981a, b). Here we focus our attention to a reduction thereof. Namely, we consider a subgroup of the Clifford group, consisting of elements g with the property

$$\begin{aligned} &(g\phi^{(1)}(P),g\phi^{(2)}(P))=(\phi^{(1)}(P)g,\phi^{(2)}(P)g)T(P),\\ &T(P)\text{: } 2\times 2 \text{ matrix}, \qquad T(P)^{\mathrm{T}}T(P^{\#})=1, \qquad \det T(P)=1. \end{aligned} \tag{3.4}$$

Henceforth we always assume (3.4). This condition implies that g commutes with $:\phi^{(1)}(P)\phi^{(1)}(P^{\#}):+:\phi^{(2)}(P)\phi^{(2)}(P^{\#}):$, and hence with $H_n^{(1)}(x)+H_n^{(2)}(x)$. The time evolution $e^{\mathrm{H}}g\,e^{-\mathrm{H}}$ of g is then invariant under the change $(n_1,n_2)\mapsto(n_1+n,n_2+n)$, $(x^{(1)},x^{(2)})\mapsto(x^{(1)}+x,x^{(2)}+x)$. Therefore, $\tau_{n_1n_2}$, $\sigma_{n_1n_2}$ and $w_{n_1n_2}^{(ij)}\exp[-\xi_{n_j}(x^{(j)},P)]$ actually depend only on the difference $n=n_1-n_2$ and $x=x^{(1)}-x^{(2)}=(x_1,x_2,\ldots)$.

We have then the following result.

(1) The combination

$$f=\tau_{n_1n_2}, \qquad f^*=\tau_{n_1+1\,n_2}, \qquad g=\sqrt{c}\,\sigma_{n_1n_2}, \qquad g^*=\sqrt{c}\,\sigma_{n_1+1\,n_2} \tag{3.5}$$

solves the bilinear Landau–Lifshitz equations (3.1) and their higher-order analogues given below.

(2) The matrix

$$\begin{aligned} W_n(x;P)=(f^*f+g^*g)^{-1/2}&\begin{pmatrix} h(P)^{-1}w_{n_1+1\,n_2}^{(11)} & -(h(P))^{-1/2}w_{n_1+1\,n_2}^{(12)} \\ -(h(P))^{-1/2}w_{n_1n_2}^{(21)} & w_{n_1n_2}^{(22)} \end{pmatrix}\\ &\times\exp\{-\tfrac{1}{2}[\xi_{n_1}(x^{(1)},P)+\xi_{n_2}(x^{(2)},P)]\} \end{aligned} \tag{3.6}$$

satisfies the linear equations of the form (1.2), and similar equations with regard to $x_3, x_4\ldots$ as well.

(3) A bilinear identity for wavefunctions

$$\sum_{\nu=1,2}\int_{\infty_+\cup\infty_-} dP\,\alpha(P)w^{(i\nu)}_{n_1n_2}(x^{(1)},x^{(2)};P)w^{(j\nu)}_{n'_1n'_2}(x^{(1)'},x^{(2)'};P^{\#})=0 \qquad (i,j=1,2) \tag{3.7}$$

holds for any $n_1, n_2, x^{(1)}, x^{(2)}, n'_1, n'_2, x^{(1)'}, x^{(2)'}$ and any function $\alpha(p)$ holomorphic except at $\infty_\pm$. Here the left-hand side signifies (minus) the sum of residues of the integrand at $\infty_\pm$.

In the definition (3.6), we have inserted the factor $(h(P))^{\pm 1/2}c^{-1/2}(z_2\pm z_1)$ so as to adjust the behaviour of $W_n(x;P)$ at $\infty_\pm$ as

$$\begin{aligned}(f^*f+g^*g)^{1/2}W_n(x;P)&=\hat{W}_n(x;P)\exp\tfrac{1}{2}\begin{pmatrix}\xi_n(x;P) & \\ & -\xi_n(x;P)\end{pmatrix},\\ \hat{W}_n(x;P)&=\mathrm{O}(1) \qquad \text{as } P\to\infty_\pm.\end{aligned} \tag{3.8}$$

Thus $W_n(x;P)$ is defined on the covering curve $\tilde{E}$ (1.3) rather than on E (1.5). Its matrix elements satisfy the symmetry

$$W_n(x;P^{\#})_{11}=W_n(x;P)_{22}, \qquad W_n(x;P^{\#})_{21}=-W_n(x;P)_{12} \tag{3.9}$$

and

$$\det W_n(x;P)=1,$$

which imply $W_n(x;P)^{\mathrm{T}}W_n(x;P^{\#})=2\times 2$ unit matrix.

In terms of $W_n(x;P)$, the bilinear identity (3.7) reads

$$\int_{\infty_+\cup\infty_-} dP\,\alpha(P)h(P)^{(n-n')/2}\begin{pmatrix}1 & \\ & (h(P))^{-1/2}\end{pmatrix}W_n(x;P)^{\mathrm{T}}W_{n'}(x';P^{\#})\begin{pmatrix}1 & \\ & (h(P))^{1/2}\end{pmatrix}=0 \tag{3.10}$$

for any (n,x), (n',x') and $\alpha(P)\in\mathbb{C}[k,\omega]$.

Example. Put

$$g=\exp\Big(\sum_{i=1}^{N}a_i\phi^{(1)}(P_i)\phi^{(2)}(P_i^{\#})\Big) \tag{3.11}$$

with $P_i\neq P_j^{\#}$ $(i\neq j)$. The requirement (3.4) is satisfied. With this choice of g, (3.5) gives an N-soliton solution

$$\begin{aligned}\left.\begin{matrix}f\\ g\end{matrix}\right\}&=\sum_{\substack{r\ \text{even}\\ \text{odd}}}\ \sum_{i_1<\ldots<i_r}c_{i_1\ldots i_r}\exp(\xi_{i_1}+\ldots+\xi_{i_r}),\\ \left.\begin{matrix}f^*\\ g^*\end{matrix}\right\}&=\sum_{\substack{r\ \text{even}\\ \text{odd}}}\ \sum_{i_1<\ldots<i_r}c_{i_1\ldots i_r}h_{i_1}\ldots h_{i_r}\exp(\xi_{i_1}+\ldots+\xi_{i_r}),\end{aligned} \tag{3.12}$$

where r ranges over $0, 1, \ldots, N$, $h_i=h(P_i)$,

$$e^{\xi_i}=-\sqrt{c}\,a_i\exp[\xi_n(x,P_i)], \qquad c_{ij}=c^{-1}K(P_iP_j)K(P_j^{\#}P_i^{\#})$$

and $c_{i_1\ldots i_r}=\Pi_{\mu<\nu}c_{i_\mu i_\nu}$. The corresponding wavefunctions (in the notation (3.8)) are given by

$$
\begin{aligned}
(\hat{W}_n(x;P))_{\substack{11\\21}} &= \sum_{\substack{r\ \text{even}\\ \text{odd}}} \sum_{i_1<\ldots<i_r} c_{i_1\ldots i_r} \frac{K(P,P_{i_1})\ldots K(P,P_{i_r})}{(ch(P))^{r/2}} \exp(\xi_{i_1}+\ldots+\xi_{i_r}),\\
(\hat{W}_n(x;P))_{\substack{22\\12}} &= \pm \sum_{\substack{r\ \text{even}\\ \text{odd}}} \sum_{i_1<\ldots<i_r} c_{i_1\ldots i_r} \frac{K(P^{\#},P_{i_1})\ldots K(P^{\#},P_{i_r})}{(ch(P^{\#}))^{r/2}} \exp(\xi_{i_1}+\ldots+\xi_{i_r}).
\end{aligned}
\tag{3.13}
$$

For instance, formulae (3.12), (3.13) for $N=2$ read respectively

$$
f = 1 + c_{12}\exp(\xi_1+\xi_2), \qquad f^* = 1 + c_{12}h_1h_2\exp(\xi_1+\xi_2),
$$

$$
g = e^{\xi_1} + e^{\xi_2}, \qquad g^* = h_1 e^{\xi_1} + h_2 e^{\xi_2}
$$

and

$$
\hat{W}_n(x;P) = \begin{pmatrix} 1 + c_{12}\dfrac{K(PP_1)K(PP_2)}{ch(P)}\exp(\xi_1+\xi_2) & -(ch(P^{\#}))^{-1/2}(K(P^{\#}P_1)e^{\xi_1} + K(P^{\#}P_2)e^{\xi_2}) \\ (ch(P))^{-1/2}(K(PP_1)e^{\xi_1} + K(PP_2)e^{\xi_2}) & 1 + c_{12}\dfrac{K(P^{\#}P_1)K(P^{\#}P_2)}{ch(P^{\#})}\exp(\xi_1+\xi_2) \end{pmatrix}.
$$

In the derivation of (3.12) and (3.13), we have used Wick's theorem and the formulae (A4), (A5) in the appendix.

Let us sketch how to prove the assertions (1)–(3). In particular, when specialised to g of the form (3.11), the following argument will provide a proof of the N-soliton formula given above.

First we show that the bilinear identity (3.7) is a simple consequence of the property (3.4) of g. Namely, using (3.3) and (3.4), we can rewrite the left-hand side of (3.7) into

$$
\sum_{\nu=1,2} \int dP\,\alpha(P)\langle \text{vac}|\phi_0^{(i)}\, e^{\mathbb{H}} g \phi^{(\nu)}(P)|\text{vac}\rangle\langle \text{vac}|\phi_0^{(j)}\, e^{\mathbb{H}'} g \phi^{(\nu)}(P^{\#})|\text{vac}\rangle \tag{3.14}
$$

with $\mathbb{H}' = H_{n_1}^{(1)}(x^{(1)'}) + H_{n_2}^{(2)}(x^{(2)'})$. On the other hand, we have, for any $\phi^{(\lambda)}(Q)$ and $\phi^{(\mu)}(R)$,

$$
\int dP\,\alpha(P)\langle \text{vac}|\phi^{(\lambda)}(Q)\phi^{(\nu)}(P)|\text{vac}\rangle\langle \text{vac}|\phi^{(\mu)}(R)\phi^{(\nu)}(P^{\#})|\text{vac}\rangle = 0,
$$

and hence

$$
\int dP\,\alpha(P)\langle a|\phi^{(\nu)}(P)|\text{vac}\rangle\langle b|\phi^{(\nu)}(P^{\#})|\text{vac}\rangle = 0
$$

for any vectors $\langle a|$, $\langle b|$. This shows the vanishing of (3.14).

In order to find bilinear equations, we exploit the expansions of $\hat{W}_n(x;P)$ at $\infty_\pm$ in terms of f, f^*, g and g^*:

$$
\hat{W}_n(x;P) = \begin{pmatrix} f\left(x_1-\dfrac{1}{k}, x_2-\dfrac{1}{2ch}, x_3-\dfrac{1}{3k^3},\ldots\right) & -g^*\left(x_1+\dfrac{1}{k}, x_2+\dfrac{1}{2ch}, x_3+\dfrac{1}{3k^3},\ldots\right) \\ g\left(x_1-\dfrac{1}{k}, x_2-\dfrac{1}{2ch}, x_3-\dfrac{1}{3k^3},\ldots\right) & f^*\left(x_1+\dfrac{1}{k}, x_2+\dfrac{1}{2ch}, x_3+\dfrac{1}{3k^3},\ldots\right) \end{pmatrix} \tag{3.15}
$$

$$\hat{W}_n(x;P^*)=$$

$$\begin{pmatrix} f^*\left(x_1+\frac{1}{k},x_2+\frac{1}{2ch},x_3+\frac{1}{3k^3},\dots\right) & -g\left(x_1-\frac{1}{k},x_2-\frac{1}{2ch},x_3-\frac{1}{3k^3},\dots\right) \\ g^*\left(x_1+\frac{1}{k},x_2+\frac{1}{2ch},x_3+\frac{1}{3k^3},\dots\right) & f\left(x_1-\frac{1}{k},x_2-\frac{1}{2ch},x_3-\frac{1}{3k^3},\dots\right) \end{pmatrix}$$

as $k\to\infty$, $\sqrt{ch}=k+\dots\to\infty$. In the language of operators, (3.15) is a direct consequence of the formulae (2.7), (2.8) for vertex operators. The bilinear identity (3.10) also enables us to derive (3.15) directly. Namely, suppose a matrix $W_n(x;P)$ satisfies (3.9) as well as

$$\det W_n(x;P)\neq 0,\qquad W_n(x;P)\exp\tfrac{1}{2}\begin{pmatrix}\xi_n(x;P) & \\ & -\xi_n(x;P)\end{pmatrix}=O(1)\quad \text{as } P\to\infty_\pm.$$

Changing $W_n(x;P)$ into

$$c(P)\begin{pmatrix}u(x) & \\ & v(x)\end{pmatrix}W_n(x;P)\quad (c(P^*)=c(P)^{-1})$$

if necessary, we can then show the existence of functions $f(x)$, $f^*(x)$, $g(x)$ and $g^*(x)$ such that (3.8), (3.9) and (3.15) are valid.

Substituting the expansions (3.15) into (3.10), we obtain the following generating functions of bilinear equations:

$$\begin{aligned}
&\mathrm{Res}_{k=\infty}\frac{dk}{\omega}a(k,\omega)G(k,D)\cosh F(y;k,D)(f^*\cdot f+g^*\cdot g)=0\\
&\qquad\qquad \text{for } a(k,\omega)=a(-k,-\omega),\\
&\mathrm{Res}_{k=\infty}\frac{dk}{\omega}a(k,\omega)G(k,D)\sinh F(y;k,D)(f^*\cdot f-g^*\cdot g)=0\\
&\qquad\qquad \text{for } a(k,\omega)=-a(-k,-\omega),\\
&\mathrm{Res}_{k=\infty}\frac{dk}{\omega}a(k,\omega)(h(k))^{\pm 1/2}G(k,D)\sinh F(y;k,D)f^*\cdot g\\
&\qquad =\mathrm{Res}_{k=\infty}\frac{dk}{\omega}a(-k,-\omega)(h(k))^{\mp 1/2}G(k,D)\sinh F(y;k,D)g^*\cdot f.
\end{aligned}\tag{3.16}$$

Here we have set

$$F(y;k,D)=\sum_{n\text{ odd}}y_n(k^n-D_n)+\sum_{n\text{ even}}y_n(c^{n/2}(h^{n/2}-h^{-n/2})-D_n),$$

$$G(k,D)=\exp\left(\sum_{n\text{ odd}}\frac{1}{nk^n}D_n+\sum_{n\text{ even}}\frac{1}{n(ch)^{n/2}}D_n\right).$$

The parameters y_n and the polynomial $a(k,\omega)\in\mathbb{C}[k,\omega]$ are arbitrary. Calculating the first few equations in (3.16), we obtain, as well as (3.1),

$$(D_2+D_1^2)(f^*\cdot f+g^*\cdot g)=0,$$

$$(2D_3+D_1^3)(f^*\cdot f+g^*\cdot g)=0,\qquad (D_3-D_1^3)(f^*\cdot f-g^*\cdot g)=0,$$

$$D_1D_2(f^*\cdot f+g^*\cdot g)=0,\qquad [D_1D_2-D_1^3+(a^2+b^2)D_1](f^*\cdot f-g^*\cdot g)=0,$$

$$(D_3 - D_1^3) f^* \cdot g = 0, \qquad (D_3 - D_1^3) g^* \cdot f = 0,$$

$$[D_1 D_2 - D_1^3 + \tfrac{1}{2}(a^2+b^2) D_1] f^* \cdot g - \tfrac{1}{2}(a^2 - b^2) D_1 g^* \cdot f = 0,$$

$$[D_1 D_2 - D_1^3 + \tfrac{1}{2}(a^2+b^2) D_1] g^* \cdot f - \tfrac{1}{2}(a^2 - b^2) D_1 f^* \cdot g = 0.$$

Existence of linear equations for $W_n(x;P)$ is also a consequence of the bilinear identity (3.10). Choose a matrix $B_j(x;P)$ so as to satisfy the conditions

(i) $\begin{pmatrix} 1 & \\ & (h(P))^{-1/2} \end{pmatrix} B_j(x;P) \begin{pmatrix} 1 & \\ & h(P))^{1/2} \end{pmatrix}$ is a polynomial of k and ω,

(ii) $B_j(x;P) - \dfrac{\partial}{\partial x_j} W_n(x;P) \cdot W_n(x;P)^{-1} = \begin{pmatrix} \mathrm{O}(k^{\mp 1}) & \mathrm{O}(1) \\ \mathrm{O}(1) & \mathrm{O}(k^{\mp 1}) \end{pmatrix}$ as $P \to \infty_\pm$.

Such a matrix always exists and is unique. The principal terms of $(\partial/\partial x_j) W_n(x;P) \cdot W_n(x;P)^{-1}$ in the sense of (ii) are explicitly given by

$$\begin{aligned}
&(f^* \cdot f + g^* \cdot g) \frac{\partial}{\partial x_j} W_n(x;P) \cdot W_n(x;P)^{-1} \\
&\qquad \equiv p_n(D,k) \frac{1}{2} \begin{pmatrix} f^* \cdot f - g^* \cdot g & 2g^* \cdot f \\ 2f^* \cdot g & -f^* \cdot f + g^* \cdot g \end{pmatrix} \qquad (P \to \infty_+), \\
&\qquad \equiv -p_n(D,-k) \frac{1}{2} \begin{pmatrix} f^* \cdot f - g^* g & 2f^* \cdot g \\ 2g^* \cdot f & -f^* \cdot f + g^* \cdot g \end{pmatrix} \qquad (P \to \infty_-),
\end{aligned}$$

modulo $\mathrm{O}(k^{-1})$, where

$$p_n(D,k) = \begin{Bmatrix} k^n \\ (ch)^{n/2} \end{Bmatrix} G(k,D) - D_n, \qquad n \begin{cases} \text{odd} \\ \text{even} \end{cases}.$$

Rewriting the coefficients of $B_j(x;P)$ in terms of S_α in (1.4), we find

$$B_1(x;P) = \sum_{\alpha=1}^{3} S_\alpha z_\alpha \sigma_\alpha,$$

$$B_2(x;P) = \sum_{\alpha,\beta,\gamma=1}^{3} \mathrm{i}\varepsilon^{\alpha\beta\gamma} S_\beta \frac{\partial S_\gamma}{\partial x_1} z_\alpha \sigma_\alpha + 2 z_1 z_2 z_3 \sum_{\alpha=1}^{3} S_\alpha z_\alpha^{-1} \sigma_\alpha,$$

$$\begin{aligned}
B_3(x;P) = {} & \sum_{\alpha=1}^{3} \frac{\partial^2 S_\alpha}{\partial x_1^2} z_\alpha \sigma_\alpha + \left[4 z_3^2 + \frac{3}{2} \left(\frac{\partial \boldsymbol{S}}{\partial x_1} \right)^2 - \tfrac{1}{2} J \boldsymbol{S} \cdot \boldsymbol{S} + \tfrac{1}{2} J_3 \right] \sum_{\alpha=1}^{3} S_\alpha z_\alpha \sigma_\alpha \\
& + \sum_{\alpha,\beta,\gamma=1}^{3} 2\mathrm{i}\varepsilon^{\alpha\beta\gamma} z_\beta z_\gamma S_\beta \frac{\partial S_\gamma}{\partial x_1} \sigma_\alpha,
\end{aligned}$$

and so forth, in agreement with (1.2). The equation of motion corresponding to the x_3 flow reads (cf Sklyanin 1979)

$$\frac{\partial}{\partial x_3} \boldsymbol{S} = \frac{\partial^3}{\partial x_1^3} \boldsymbol{S} + \frac{3}{2} \left[\left(\frac{\partial \boldsymbol{S}}{\partial x_1} \right)^2 - J \boldsymbol{S} \cdot \boldsymbol{S} + J_3 \right] \frac{\partial \boldsymbol{S}}{\partial x_1} + 3 \left(\frac{\partial \boldsymbol{S}}{\partial x_1} \cdot \frac{\partial^2 \boldsymbol{S}}{\partial x_1^2} \right) \boldsymbol{S}.$$

Differentiating (3.10) and using (i), we get

$$\int \mathrm{d}P\, \alpha(P) \begin{pmatrix} 1 & \\ & (h(P))^{-1/2} \end{pmatrix} \left(\frac{\partial}{\partial x_j} W_n(x;P) \cdot W_n(x;P)^{-1} - B_j(x;P) \right) \begin{pmatrix} 1 & \\ & (h(P))^{1/2} \end{pmatrix} = 0$$

for any $\alpha(P) \in \mathbb{C}[k, \omega]$. In view of the growth condition (ii), we thus conclude that

$$\partial W_n(x;P)/\partial x_j = B_j(x;P) W_n(x;P).$$

This provides the Lax equations for the Landau–Lifshitz hierarchy.

From the operator representation (3.2), (3.3), we see that quadratic elements of the following form act as infinitesimal transformations of solutions:

$$\begin{aligned}\int_{\infty_+} \mathrm{d}P\, \alpha(P) :\phi^{(1)}(P)\phi^{(1)}(P^{\#}) - \phi^{(2)}(P)\phi^{(2)}(P^{\#}): + c \cdot 1 \\ + \int_{\infty_+} \mathrm{d}P\, (\beta(P):\phi^{(1)}(P)\phi^{(2)}(P^{\#}): + \gamma(P):\phi^{(2)}(P)\phi^{(1)}(P^{\#}):)\end{aligned} \tag{3.17}$$

where $\alpha(P)$, $\beta(P)$, $\gamma(P) \in \mathbb{C}[k, k^{-1}, \omega]$ and $c \in \mathbb{C}$. In view of (2.10) and (2.11), the first line of (3.17) corresponds to generators of the Heisenberg algebra $\partial/\partial x_n^{(1)} - \partial/\partial x_n^{(2)}$, $x_n^{(1)} - x_n^{(2)}$ and 1. On the other hand, (3.11) shows that $\phi^{(1)}(P)\phi^{(2)}(P^{\#})$ serves as the infinitesimal Bäcklund transformation which sends an N-soliton solution to an $(N+1)$-soliton solution. This is the meaning of the second line of (3.17). We can represent (3.17) in the basis $\phi^{(j)}(P)$ by a 2×2 matrix

$$A(P) = \begin{pmatrix} \alpha(P) & \beta(P) \\ \gamma(P) & -\alpha(P) \end{pmatrix} \in \mathrm{sl}(2, \mathbb{C}[k, k^{-1}, \omega])$$

and a scalar $c \in \mathbb{C}$. The bracket of two such elements is then given by

$$[A(P) + c \cdot 1, A'(P) + c' \cdot 1] = [A(P), A'(P)] + c(A, A') \cdot 1,$$

where the bracket on the right-hand side is the usual matrix commutator, and

$$c(A, A') = \mathrm{Res}_{\infty_+} \mathrm{Tr}\, A(P)\, \mathrm{d}A'(P). \tag{3.18}$$

In this sense, the Landau–Lifshitz hierarchy allows as infinitesimal transformations the Lie algebra

$$\mathrm{sl}(2, \mathbb{C}[k, k^{-1}, \omega]) \oplus \mathbb{C} \cdot 1$$

whose rule of central extension is given by (3.18).

4. Isotropic limit

As was pointed out by Sklyanin (1979), the equation (1.1) is a classical and continuous limit of the XYZ model. In the XXX case ($J_1 = J_2 = J_3$) the elliptic curve degenerates to a rational curve. The bilinear equations for this particular case are obtained from (3.16) by setting $a = b = 0$ (Hirota 1982). In this section we show how to scale soliton solutions and the corresponding wavefunctions in the limit $a, b \to 0$. Since $c \to 0$ in (3.5), this is not definitely obvious.

We start from the $M+N$ soliton with

$$g = \exp\left(\sqrt{c} \sum_{j=1}^{M} \frac{a_j}{k_j^2} \phi^{(1)}(P_j)\phi^{(2)}(P_j^{\#}) + \sqrt{c}^{-1} \sum_{j=M+1}^{M+N} a_j \phi^{(1)}(P_j^{\#})\phi^{(2)}(P_j)\right)$$

in (3.2) and (3.3). Here we choose P_j ($j = 1, \ldots, M+N$) to be near ∞_+. From (2.1), the following limits for P near ∞_+ exist:

$$\lim_{a,b\to 0} \sqrt{c}\,\phi(P)=k^2\psi(k), \qquad \psi(k)=\sum_{l\in \mathbf{Z}} k^l\psi_l,$$

$$\lim_{a,b\to 0} \sqrt{c}^{-1}\phi(P^{*})=k^{-1}\psi^*(k), \qquad \psi^*(k)=\sum_{l\in \mathbf{Z}} k^{-l}\psi_l^*.$$

By using these formulae we have

$$\begin{aligned}
\bar f &\underset{\text{def}}{=} \lim_{a,b\to 0}\tau_{00}=\langle \mathrm{vac}|\psi_0^{(2)*}\,\mathrm{e}^{\bar H(x)}\bar{g}\psi_0^{(2)}|\mathrm{vac}\rangle,\\
\bar f^* &\underset{\text{def}}{=} \lim_{a,b\to 0}\tau_{10}=\langle \mathrm{vac}|\psi_{-1}^{(2)}\,\mathrm{e}^{\bar H(x)}\bar{g}\psi_{-1}^{(2)*}|\mathrm{vac}\rangle,\\
\bar g &\underset{\text{def}}{=} \lim_{a,b\to 0}\sqrt{c}\,\sigma_{00}=-\langle \mathrm{vac}|\psi_{-1}^{(1)}\,\mathrm{e}^{\bar H(x)}\bar{g}\psi_{-1}^{(2)*}|\mathrm{vac}\rangle,\\
\bar g^* &\underset{\text{def}}{=} \lim_{a,b\to 0}\sqrt{c}\,\sigma_{10}=-\langle \mathrm{vac}|\psi_0^{*(1)}\,\mathrm{e}^{\bar H(x)}\bar{g}\psi_0^{(2)}|\mathrm{vac}\rangle,
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\bar W(x) &= \lim_{a,b\to 0} W_0(x;P)\\
&=\frac{\exp[-\frac{1}{2}\xi(x^{(1)}+x^{(2)},k)]}{(\bar f^*\bar f+\bar g^*\bar g)^{1/2}}\\
&\times\begin{pmatrix} -\langle \mathrm{vac}|\psi_0^{(1)*}\psi_0^{(2)*}\,\mathrm{e}^{\bar H(x)}\psi^{(1)}(k)\bar g\psi_0^{(2)}|\mathrm{vac}\rangle & \langle \mathrm{vac}|\psi_0^{(1)*}\psi_0^{(2)*}\,\mathrm{e}^{\bar H(x)}\psi^{(2)}(k)\bar g\psi_0^{(2)}|\mathrm{vac}\rangle\\ -\langle \mathrm{vac}|\mathrm{e}^{\bar H(x)}\psi^{(1)}(k)\bar g\psi_{-1}^{(2)*}|\mathrm{vac}\rangle k & \langle \mathrm{vac}|\mathrm{e}^{\bar H(x)}\psi^{(2)}(k)\bar g\psi_{-1}^{(2)*}|\mathrm{vac}\rangle k\end{pmatrix}.
\end{aligned}$$

Here we have set (cf Date *et al* 1981a)

$$\bar H(x)=\sum_{\alpha=1,2}\sum_{l\geqslant 1}\sum_{n\in\mathbf{Z}} x_l^{(\alpha)}\psi_n^{(\alpha)}\psi_{n+l}^{(\alpha)*},$$

$$\bar g=\exp\left(\sum_{j=1}^{M} a_j\psi^{(1)}(k_j)\psi^{(2)*}(k_j)+\sum_{j=M+1}^{M+N} a_j\psi^{(1)*}(k_j)\psi^{(2)}(k_j)\right),$$

$$\xi(x,k)=\sum_{l\geqslant 1}k^l x_l.$$

It is not difficult to obtain explicit forms of soliton solutions from (4.1) by using Wick's theorem. For instance, the case $M=N=1$ reads

$$\bar f=1-[k_2^2/(k_1-k_2)^2]\exp(\xi_1-\xi_2), \qquad \bar f^*=1-[k_1^2/[(k_1-k_2)^2]\exp(\xi_1-\xi_2),$$

$$\bar g=-\mathrm{e}^{-\xi_2}, \qquad \bar g^*=-\mathrm{e}^{\xi_1},$$

where

$$\mathrm{e}^{-\xi_2}=a_2\exp[-\xi(x^{(1)}-x^{(2)},k_2)], \qquad \mathrm{e}^{\xi_1}=a_1\exp[\xi(x^{(1)}-x^{(2)},k_1)].$$

5. Quasi-periodic solution

In this section we construct quasi-periodic solutions of the Landau–Lifshitz equation by constructing wavefunctions which satisfy the bilinear identity for the Landau–Lifshitz equation (3.10).

Let C be a non-singular algebraic curve of genus g with a double covering map $\pi : C \to E$ not ramifying at $\infty_\pm \in E$. We assume further that C admits a fixed-point free involution ι compatible with the involution $\#$ on $E : \# \circ \pi = \pi \circ \iota$. Let $\bar{C}$ be the quotient of C by ι and $\pi_1 : C \to \bar{C}$ be the projection. By the Riemann–Hurwitz formula, we have $g = 2\bar{g} - 1$, where $\bar{g}$ is the genus of $\bar{C}$. Let $\pi^{-1}(\infty_\pm) = \infty_\pm^{(1)}, \infty_\pm^{(2)}$. As local parameters around $\infty_\pm^{(\alpha)}$, $\alpha = 1, 2$, we take $(k \circ \pi)^{-1} = \tilde{k}^{-1}$, where k is the rational function on E.

We want to construct functions $v_i(x, P)$, $i = 1, 2$, $P \in C$ with the following properties:

(i) $v_i(x, P)$ are meromorphic on $C - \{\infty_\pm^{(1)}, \infty_\pm^{(2)}\}$ with pole divisors D_i independent of x;

(ii) at $\infty_\pm^{(\alpha)}$ $(\alpha = 1, 2)$, $v_i(x, P)$ $(i = 1, 2)$ behave like

$$v_i(x, P) = (\mathrm{O}(\tilde{k}^{l_{i\pm}^{(1)}})) \exp[\xi(x, P)] \qquad \text{at } \infty_\pm^{(1)},$$

$$v_i(x, P) = (\mathrm{O}(\tilde{k}^{l_{i\pm}^{(2)}})) \exp[-\xi(x, P)] \qquad \text{at } \infty_\pm^{(2)},\ l_{i+}^{(\alpha)} = -l_{i-}^{(\alpha)} \in \mathbb{Z},$$

where $\xi(x, P) = \sum_{j\ \mathrm{odd} \geq 1} x_j \tilde{k}^j + \sum_{j\ \mathrm{even} > 1} x_j (\tilde{h}^{j/2} - \tilde{h}^{-j/2}) c^{j/2}$, $\tilde{h} = h \circ \pi$;

(iii) for a rational one-form θ on C such that $\iota^* \theta = \theta$ and having no poles and zeros at $\infty_\pm^{(\alpha)}$, $v_i(x, P)$ satisfy the bilinear identity

$$\int_{\infty_\pm^{(1)} \cup \infty_\pm^{(2)}} v_i(x, P) v_j(x', \iota P) \theta(P) = 0 \qquad \forall i, j, \forall x, x'.$$

We assume that functions $v_i(x, P)$ satisfying the above conditions (i), (ii), (iii) exist. We put

$$f = \theta / \pi^* \mathrm{d}P, \qquad f \circ \iota = f$$

and define functions $V_{ij}(x, P)$ in the neighbourhood of $\infty_\pm$ as

$$V_{ij}(x, P) = v_i(x, \tilde{P}_j)(f(\tilde{P}_j))^{1/2}, \qquad \pi(\tilde{P}_j) = P,\ \tilde{P}_j \text{ near } \infty_\pm.$$

We set

$$V(x, P) = \begin{pmatrix} V_{11}(x, P) & V_{12}(x, P) \\ V_{21}(x, P) & V_{22}(x, P) \end{pmatrix}.$$

Then the function $V(x, P)$ satisfies the bilinear identity on E

$$\int_{\infty \pm} V(x, P)^{\mathrm{T}} V(x', \iota P) \, \mathrm{d}P = 0 \tag{5.1}$$

in view of the invariance of f with respect to ι.

Now let us examine the conditions on the location of the pole divisor D_i imposed by the above conditions (ii), (iii). First, from (iii) we must have

$$D_i + \iota D_j < (\theta), \qquad i, j = 1, 2.$$

Hence in the generic case, we have $D_1 = D_2 = D$, $\deg D = g - 1$, and (iii) is rephrased as

(iii)′ $\qquad D + \iota D = (\theta), \qquad \iota^* \theta = \theta.$

If we denote by $D_i(x)$ the zero divisors of $v_i(x, P)$, the condition (ii) implies that the images of $D_i(x) - D$ in the Jacobian variety $\mathbb{J}(C)$ of C move in the Prym variety $\mathbb{P}$ (= the odd part of $\mathbb{J}(C)$ with respect to ι). In other words, the translate of $\mathbb{P}$ by the image of D in $\mathbb{J}(C)$ should lie in the theta divisor (Θ) of $\mathbb{J}(C)$.

Summarising, we must have

(a) $D + \iota D = (\theta), \qquad \iota^*\theta = \theta, \qquad D > 0.$

(b) $\mathbb{P}$ + image of $D \subset (\Theta)$ in $\mathbb{J}(C)$.

Conversely, the above conditions (a), (b) are sufficient for the existence of $v_i(x, P)$ satisfying (i), (ii), (iii)′.

On the other hand, the following is known in the theory of algebraic curves with a fixed-point free involution (cf Fay 1973).

The Prym variety $\mathbb{P}$ is a principally polarised abelian variety.

We put

$$A_{\text{even}} = \{D\colon \text{divisor on } C, D > 0, \deg D = g - 1, \pi_{1*}(D) = K_{\bar{C}}, i(D) = \text{even}\},$$

$$A_{\text{odd}} = \{D\colon \text{divisor on } C, D > 0, \deg D = g - 1, \pi_{1*}(D) = K_{\bar{C}}, i(D) = \text{odd}\},$$

where $\pi_{1*}(D)$ is the image of D on $\bar{C}$, $K_{\bar{C}}$ is the canonical divisor of $\bar{C}$ and $i(D)$ is the irregularity of D (= the dimension of the vector space of one-forms on C which have zeros at D). Then we have

$$\mathbb{P} = \mathbb{P}_{\text{odd}} \cup \mathbb{P}_{\text{even}}$$

where

$$\mathbb{P}_{\text{odd}} = \{\text{image of } A_{\text{odd}} \text{ in } \mathbb{J}(C)\} + \Delta + \delta_1,$$

$$\mathbb{P}_{\text{even}} = \{\text{image of } A_{\text{even}} \text{ in } \mathbb{J}(C)\} + \Delta.$$

Here Δ is some constant depending on the choice of the basis of $H_1(C, \mathbb{Z})$, $H^0(C, \Omega^1)$ and the base point of the map $C \to \mathbb{J}(C)$, and δ_1 is some half-period in $\mathbb{J}(C)$. Further, $\mathbb{P}_{\text{even}}$ is the zero divisor of the theta function $\vartheta_{\mathbb{P}}$ associated with the Prym variety $\mathbb{P}$.

Now we determine the choice of our divisor D. First of all, a divisor D satisfying (a) must be an element of either A_{even} or A_{odd}. The above relation among $\mathbb{P}$, A_{odd} and A_{even} shows that for either choice of D the condition (b) is also satisfied by suitably modifying the behaviour of v_i at $\infty_\pm^{(\alpha)}$. More explicitly, for $D \in A_{\text{odd}}$ (resp A_{even}), we take $l_{1+}^{(\alpha)} = 0$, $l_{2+}^{(\alpha)} = 1$, $\alpha = 1, 2$ (resp $l_{1+}^{(1)} = 0$, $l_{2+}^{(1)} = 1$, $l_{1+}^{(2)}, l_{2+}^{(2)} = 0$) in (ii). For such a choice of $l_{i\pm}^{(\alpha)}$, the bilinear identity (5.1) on E is the same as in the previous section (3.10). Thus we have constructed a solution of the Landau–Lifshitz equation. By a standard technique, we can express the functions $v_i(x, P)$ in terms of the theta function $\vartheta_{\mathbb{P}}$ mentioned above and abelian integrals on C. The corresponding τ function is then shown to coincide with $\vartheta_{\mathbb{P}}$ (cf Date *et al* 1982b).

Finally, we must note that another construction of quasi-periodic solutions of the Landau–Lifshitz equation was done by Cherednik 1981. The similarity between his construction and ours is yet to be clarified.

Acknowledgment

The authors are grateful to R Hirota for his introduction to the Landau–Lifshitz equation and his interest in this work.

Appendix

Here we summarise several formulae related to the Cauchy kernel

$$K(P, P') = \tfrac{1}{2}(\omega + k^2 - \omega' - k'^2)/(k + k')$$

and

$$h(P)=(\omega+k^2-a^2)/(\omega-k^2+a^2):$$

$$K(P,P')=-K(P',P), \tag{A1}$$

$$K(P,P')=h(P)h(P')K(P^{\#},P'^{\#}), \qquad h(P)=h(P^{\#})^{-1},$$

$$K(P,P')K(P,P'^{\#})=c\quad h(P), \qquad \text{where } c=\tfrac{1}{4}(a^2-b^2),$$

$$K(P,P')=c\frac{h(P)-h(P')}{k+k'}=\frac{k-k'}{1-h(P)^{-1}h(P')^{-1}}, \tag{A2}$$

$$h(P)^{\pm 1}=(2c)^{-1}(\pm\omega+k^2-\tfrac{1}{2}(a^2+b^2)),$$

$$c(P,P')\equiv\frac{1}{c}K(P,P')K(P'^{\#},P^{\#})=\frac{k-k'}{k+k'}\,\frac{h(P)-h(P')}{1-h(P)h(P')}. \tag{A3}$$

For N even,

$$\text{Pfaffian}(K(P_i,P_j))_{1\le i,j\le N}$$

$$=c^{-N(N-2)/4}\left(\prod_{i=1}^{N}h(P_i)\right)^{-(N-2)/2}\prod_{1\le i<j\le N}K(P_i,P_j). \tag{A4}$$

For N odd,

$$\sum_{\nu=1}^{N}(-)^{\nu-1}\,\text{Pfaffian}(K(P_i,P_j))_{\substack{1\le i,j\le N\\ i,j\ne\nu}}$$

$$=c^{-(N-1)^2/4}\left(\prod_{i=1}^{N}h(P_i)\right)^{-(N-1)/2}\prod_{1\le i<j\le N}K(P_i,P_j), \tag{A5}$$

$$\sum_{\nu=1}^{N}(-)^{\nu-1}h(P_\nu)\,\text{Pfaffian}(K(P_i,P_j))_{\substack{1\le i,j\le N\\ i,j\ne\nu}}$$

$$=c^{-(N-1)^2/4}\left(\prod_{i=1}^{N}h(P_i)\right)^{-(N-3)/2}\prod_{1\le i<j\le N}K(P_i,P_j). \tag{A6}$$

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